Similarity solutions in one-dimensional relativistic gas dynamics

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The Liang equation which describes the one-dimensional motion of a relativistic fluid in the hodograph plane is treated with the methods of Lie group analysis. In particular, it is shown that this equation admits Lie symmetries if the sound speed satisfies a differential condition. A certain number of similarity solutions are also given in correspondence to some specific determinations of the sound speed which are of physical interest. [S1063-651X(97)15808-1]

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I. INTRODUCTION

Finding analytical solutions for the equations describing the motion of a relativistic fluid is, in general, difficult due to their nonlinear character. However, when one considers a one-dimensional flow, this problem can be overcome by carrying out a transformation which interchanges the roles of dependent and independent variables. This procedure leads to a linear equation, the hodograph equation of the flow, which was derived by Liang [1]. The problem remains of finding explicit solutions to this equation which contains an ''arbitrary element,'' i.e., a variable coefficient deriving from the particular equation of state choosen to characterize the fluid. ''Arbitrary elements'' are functions or variable parameters whose form is not known *a priori* and can be assigned freely on the grounds of physical hypotheses about the nature of the medium under consideration. In our case, the use of one equation of state rather than another, is a problem in itself, in that it should be compatible with the general principles of relativity (e.g., causality principle). It is well known that the classical equations of state are not only inappropriate to physically describe the behavior of a relativistic fluid, but also the equations governing the motion can assume a complicated structure from the point of view of the integration. Equations of state coming from genuine relativistic considerations, though more complicated at first sight, can, on the contrary, be enormously simplifying, in that they endow the equation with a high degree of symmetry. To express this concept, mathematicians have formulated a ''simplicity criterion'' which states that the arbitrary elements of an equation must be choosen in such a way as to make it somewhat easy to obtain explicit solutions [2]. The ''simplicity criterion'' is a natural product of group analysis, i.e., the mathematical technique whose object is the determination of the complete invariance group admitted by a differential equation or a system of differential equations.

The theory of group analysis was discovered and applied by S. Lie in the nineteenth century, but only in the last decades has it become a common tool for both mathematicians and physicists. The method consists of looking for the infinitesimal generators of a group of point transformations which leave the equation under study invariant. An important point of the Lie theory is that the conditions for an equation to admit a group of transformations are represented by a set of linear equations, the so-called ''determining equations,'' which are usually completely solvable. Having once found the groups of transformations, one can obtain a number of interesting results among which is the possibility to reduce a partial differential equation in two independent variables into an ordinary differential equation in one independent variable, which is generally ammenable to one of the classical equations. These particular solutions are called ''similarity solutions'' [3]. When the equation contains ''arbitrary elements,'' the theory gives a differential condition for them. This additional tie corresponds, in our case, to classes of equations of state, among which one can select those physically motivated.

This paper is organized as follows: In Sec. II, we derive the linear equation which describes the relativistic flow in the hodograph plane. In Sec. III we carry out its group analysis. In Sec. IV we consider some specific equations of state. In Sec. V we find the corresponding similarity solutions.

II. THE RELATIVISTIC FLUID

When the relativistic fluid is assumed to be perfect, namely, with zero viscosity and thermal conductivity, then it is described by the energy-momentum tensor [4]

$$T^{ab}=(p+\mu)u^a u^b - p g^{ab} \quad (a,b=0,1,2,3), \tag{1}$$

where p and μ are, respectively, the hydrodynamic pressure and the total energy density measured in a frame in which the fluid is at rest, u^a is the normalized four-velocity oriented towards the future, and g^{ab} is the metric tensor, so that $g^{ab}u^a u^b=1$. For simplicity, units are chosen which make the velocity of light equal to unity $(c=1)$.

The equations governing the motion of such a fluid are the equation expressing the conservation of the energy-momentum tensor and the equation expressing the conservation of material density, which, respectively, read

$$\nabla_a T^{ab}=0, \quad \nabla_a(\rho u^a)=0 \tag{2}$$

where ∇_a is the covariant derivative and ρ the rest mass density. It is also useful to introduce the specific internal energy e and the relativistic enthalpy density w defined, respectively, by

$$e=\frac{\mu}{\rho}-1, \quad w=\int \frac{dp}{p+\mu}\,. \tag{3}$$

The first law of thermodynamics reads $de = Tds - pd(1/\rho)$, with T as the absolute temperature and s the specific entropy. Making the assumption that the flow is isentropic one has simply

$$de = (p/\rho^2)d\rho. \quad (4)$$

To close system (1)–(3) we need an equation of state relating the quantities μ, p, and s, i.e., $\mu = \mu(p,s)$. As the speed of sound in the rest frame of the fluid is defined by $c_s = \sqrt{\partial p/\partial \mu}$, we have

$$\frac{dw}{d\rho} = \frac{c_s^2}{\rho}. \quad (5)$$

Let (R_4, g) be a given Minkowski space time, where g is the flat metric, $\mathbf{x}$ a point belonging to R_4, and x^a ($a = 0,1,2,3$) pseudo-Cartesian coordinates of $\mathbf{x}$, $\mathrm{sgn} g^{ab} = (+1, -1, -1, -1)$.

As we confined our study to the one-dimensional flow within the flat space R_4, we chose an orthogonalized couple of constant congruences $\{\xi^a, \zeta^a\}$, such that

$$g_{ab}\xi^a\xi^b = 1, \quad g_{ab}\zeta^a\zeta^b = -1,$$

$$g_{ab}\xi^a\xi^b = 0, \quad \partial_b\xi^a = \partial_b\zeta^a = 0, \quad (6)$$

which imply

$$t = x^a\xi_a, \quad x = x^a\zeta_a, \quad \frac{\partial}{\partial x^a} = \xi_a\frac{\partial}{\partial t} + \zeta_a\frac{\partial}{\partial x}, \quad (7)$$

while the four velocity can be written as

$$u^a = \gamma\{\xi^a + \nu\zeta^a\}, \quad (8)$$

where $\gamma = 1/\sqrt{1-\nu^2}$ is the Lorentz factor and ν the relative velocity.

With the foregoing hypotheses, the equations of the relativistic one-dimensional flow in three-vector notation read

$$\frac{\partial w}{\partial t} + c_s^2\tanh\theta\frac{\partial\theta}{\partial t} + \tanh\theta\frac{\partial w}{\partial x} + c_s^2\frac{\partial\theta}{\partial x} = 0, \quad (9)$$

$$\frac{\partial\theta}{\partial t} + \tanh\theta\frac{\partial w}{\partial t} + \tanh\theta\frac{\partial\theta}{\partial x} + \frac{\partial w}{\partial x} = 0, \quad (10)$$

where

$$\theta = \mathrm{arctanh}\,\nu \quad \text{and} \quad c_s = c_s(w).$$

Liang [1] has shown that the nonlinear system (9) and (10) can be transformed into a linear second order partial differential equation by using the hodograph method, which essentially consists of interchanging of the roles of dependent and independent variables $\{w(x,t), \theta(x,t)\} \Rightarrow \{x(w,\theta), t(w,\theta)\}$. This can be easily carried out by writing the partial derivatives in Eqs. (9) and (10) in the form of Jacobians, i.e.,

$$\frac{\partial(w,x)}{\partial(t,x)} + c_s^2\tanh\theta\frac{\partial(\theta,x)}{\partial(t,x)} + \tanh\theta\frac{\partial(w,t)}{\partial(x,t)} + c_s^2\frac{\partial(\theta,t)}{\partial(x,t)} = 0,$$

$$\frac{\partial(\theta,x)}{\partial(t,x)} + \tanh\theta\frac{\partial(w,x)}{\partial(t,x)} + \tanh\theta\frac{\partial(\theta,t)}{\partial(x,t)} + \frac{\partial(w,t)}{\partial(x,t)} = 0,$$

whereupon the multiplication by

$$\frac{\partial(t,x)}{\partial(w,\theta)} \equiv \frac{\partial t}{\partial w}\frac{\partial x}{\partial\theta} - \frac{\partial t}{\partial\theta}\frac{\partial x}{\partial w} \neq 0,$$

yields the linear partial differential equations sought in the unknowns $x(w,\theta)$ and $t(w,\theta)$. Here

$$\frac{\partial x}{\partial\theta} - c_s^2\tanh\theta\frac{\partial x}{\partial w} - \tanh\theta\frac{\partial t}{\partial\theta} + c_s^2\frac{\partial t}{\partial w} = 0, \quad (11)$$

$$\frac{\partial x}{\partial w} - \tanh\theta\frac{\partial x}{\partial\theta} - \tanh\theta\frac{\partial t}{\partial w} + \frac{\partial t}{\partial\theta} = 0. \quad (12)$$

It is easy to check that Eq. (12) is identically satisfied if one writes t and x in terms of a potential function $\Psi(w,\theta)$

$$t = e^{-w}(\Psi_w\cosh\theta - \Psi_\theta\sinh\theta),$$

$$x = e^{-w}(\Psi_w\sinh\theta - \Psi_\theta\cosh\theta),$$

$$\Psi_w \equiv \frac{\partial\Psi}{\partial w},$$

$$\Psi_\theta \equiv \frac{\partial\Psi}{\partial\theta}, \quad (13)$$

afterwards the insertion of Eq. (13) into Eq. (11) yields the Liang equation [1]

$$c_s^2\Psi_{ww} + (1 - c_s^2)\Psi_w - \Psi_{\theta\theta} = 0, \quad c_s = c_s(w). \quad (14)$$

So that the problem represented by Eqs. (9) and (10) is reduced to the search for solutions to the linear equation (14), which contains the arbitrary element $c_s = c_s(w)$ depending on the equation of state characterizing the relativistic fluid. In Sec. III we shall find the determinations of $c_s = c_s(w)$ which endow Eq. (14) with Lie symmetries.

III. THE GROUP ANALYSIS OF EQ. (14)

Background and procedures of the modern Lie group theory are well described in literature [2,3,5,6]. The paper by Bluman and Kumei [7] is particularly useful for our analysis.

Essentially we require the invariance of Eq. (14) under the infinitesimal transformations

$$\hat{w} = w + \varepsilon W(w,\theta,\Psi) + O(\varepsilon^2), \quad (15)$$

$$\hat{\theta} = \theta + \varepsilon\Theta(w,\theta,\Psi) + O(\varepsilon^2), \quad (16)$$

$$\hat{\Psi} = \Psi + \varepsilon g(w,\theta)\Psi + O(\varepsilon^2). \quad (17)$$

The application of the Lie conditions for the invariance of Eq. (14) yields an overdetermined set of linear equations in the unknowns W, Θ, and g, the so-called ‘‘determining equations’’

$$qg_{ww} + (1-q)g_w - g_{\theta\theta} = 0, \quad q = c_s^2(w), \quad (18)$$

$$q\Theta_{ww}+(1-q)\Theta_w-\Theta_{\theta\theta}+2g_\theta=0, \tag{19}$$

$$qW_{ww}-(1-q)W_w-W_{\theta\theta}+[q'(w)/q]W-2qg_w=0, \tag{20}$$

$$q\Theta_w-W_\theta=0, \tag{21}$$

$$2q(\Theta_\theta-W_w)+q'(w)W=0, \tag{22}$$

$$W_{\Psi\Psi}=\Theta_{\Psi\Psi}=W_\Psi=\Theta_\Psi=0. \tag{23}$$

Eliminating Θ from Eqs. (21) and (22) we obtain the relationship

$$\left\{W\left[\ln\left(\frac{W}{\sqrt{q}}\right)\right]_w\right\}_w=\frac{W_{\theta\theta}}{q}, \tag{24}$$

whereupon it is easy to deduce from Eqs. (19) and (20) that

$$g_\theta=\frac{1}{2}\left\{\left(\frac{q'(w)}{2q}-\frac{1}{q}+1\right)W\right\}_\theta,$$

$$g_w=\frac{1}{2}\left\{\left(\frac{q'(w)}{2q}-\frac{1}{q}+1\right)W\right\}_w \tag{25}$$

we have, therefore,

$$g=\frac{Q}{2}W+\text{const}, \tag{26}$$

where

$$Q=\frac{q'(w)}{2q}-\frac{1}{q}+1. \tag{27}$$

Making g satisfy Eq. (18), we obtain

$$\left\{\left(Q'(w)+\frac{Q^2}{2}+\frac{Q}{q}-Q\right)W^2\right\}_w=0, \tag{28}$$

from which we see that W has the form

$$W=\frac{f(\theta)}{\sqrt{F(w)}}, \tag{29}$$

where $f(\theta)$ is an arbitrary function for the time being, and

$$F(w)=-\left[Q'(w)+\frac{Q^2}{2}+\frac{Q}{q}-Q\right]. \tag{30}$$

Before going on to the determination of the infinitesimal generators, we notice that Eq. (14) admits an infinite parameter group if $F(w)=0$, i.e.,

$$Q'(w)+\frac{Q^2}{2}+\frac{Q}{q}-Q=\frac{1}{2q^2}\{qq''(w)-\tfrac{3}{4}[q'(w)]^2+2q'(w)-(1-q)^2\}=0. \tag{31}$$

On the other hand, for $F(w)\neq 0$, the insertion of Eq. (29) into Eq. (24) yields

$$\frac{q}{2}\sqrt{F}\frac{d}{dw}\left\{\frac{1}{\sqrt{F}}\frac{d}{dw}\ln(qF)\right\}=-\frac{f''(\theta)}{f}=\omega^2=\text{const}, \tag{32}$$

while from Eqs. (26), (29), and (18) we deduce

$$q\frac{d^2}{dw^2}\left(\frac{Q}{\sqrt{F}}\right)+(1-q)\frac{d}{dw}\left(\frac{Q}{\sqrt{F}}\right)+\omega^2\left(\frac{Q}{\sqrt{F}}\right)=0, \tag{33}$$

so that if, and only if, $q=c_s^2(w)$ satisfies Eqs. (32) and (33), Eq. (14) admits a Lie group of symmetries.

For $\omega=0$, both Eqs. (32) and (33) are satisfied when $qF=k=\text{const}$, i.e.,

$$qq''(w)-\tfrac{3}{4}[q'(w)]^2+q'(w)-(1-q)^2+2kq=0, \tag{34}$$

and

$$f''(\theta)=0. \tag{35}$$

We now derive the infinitesimal generator corresponding to $qF=\text{const}$. Equations (26) and (29) yield, respectively,

$$g\Psi=\left(\frac{Q}{2}W+C\right)\Psi,\quad W=\frac{f(\theta)}{\sqrt{F}}=\sqrt{q}(A\theta+B);$$

while from Eqs. (21) and (22) we obtain

$$\Theta_w=\frac{W_\theta}{q},\quad \Theta_\theta=W_w-\frac{q'(w)}{2q}W=0;$$

hence

$$\Theta=A\int\frac{dw}{\sqrt{q}}+D,$$

where the arbitrary constants A, B, C, and D are the group parameters.

In Sec. IV we shall discuss the equations of state corresponding to condition (31) (which gives an infinite group) and to condition (34).

IV. SPECIFIC EQUATIONS OF STATE

(a) When Eq. (31) holds true, Eq. (14) admits an infinite group of transformations and maps into the wave equation

$$X_{\varphi\varphi}-X_{\theta\theta}=0, \tag{36}$$

where

$$X=\varphi(w)\Psi(w,\theta)\quad\text{and}\quad \varphi=\frac{4\sqrt{q}}{q'(w)-2(1-q)},$$

on condition that $q'(w)-2(1-q)\neq 0$.

On the other hand, let us suppose that $Q=0$, i.e.,

$$q'(w)-2(1-q)=0. \tag{37}$$

Keeping in mind that $q=c_s^2=dp/d\mu$ and the definition (3) of enthalpy, we can write Eq. (37) as

$$(\mu+p)\mu_{pp}-2\mu_p(1-\mu_p)=0, \tag{38}$$

hence we obtain the equation of state in parametric form

$$\mu(\tau)=a(\sinh\tau+\tau)+b, \quad p(\tau)=a(\sinh\tau-\tau)-b \tag{39}$$

where a and b are two constants. The speed of sound is $c_s=\tanh(\tau/2)\leqslant 1$. For $\tau\to\infty$ and $b=0$ one has $\mu=p$ and $c_s=1$. The latter characterizes an incompressible relativistic fluid. Both equations of state are compatible with the causality principle.

The transformation taking Eq. (14) into Eq. (36) is, this time,

$$X=\Psi, \quad \varphi=\ln\{e^w+\sqrt{e^{2w}-1}\}. \tag{40}$$

It is worth mentioning that, with the foregoing equations of state, even the nonlinear system (9) and (10) undergoes a remarkable simplification, in that it can be written in the form of two independent equations

$$U_t+UU_x=0 \quad \text{and} \quad V_t+VV_x=0, \tag{41}$$

if one choose as dependent variables $U=\tanh(\theta+\varphi)$ and $V=\tanh(\theta-\varphi)$ [8].

(b) We now consider the case in which Eq. (14) is invariant under a four-parameter Lie group of point transformations, this happens when $q(w)$ satisfies Eq. (34). Here we give some solutions.

(i) $q=(\gamma-1)=\text{const}$, which corresponds to the equation of state $p=(\gamma-1)\mu$ for a barotropic fluid $(1<\gamma<2)$. In this case $k=(\gamma-2)^2/2(\gamma-1)$, in particular, for $k=2/3$ one has $p=\mu/3$, which characterizes a three-dimensional incompressible relativistic flow [9].

(ii) Another interesting solution corresponding to $k=2/3$ is obtained by observing that Eq. (34) is satisfied by

$$q'(w)=2(\tfrac{1}{3}-q), \tag{42}$$

which, in terms of the pressure p and energy density μ, reads

$$(\mu+p)\mu_{pp}-2\mu_p(1-\mu_p/3), \tag{43}$$

whose solution in parametric form is

$$\mu(\tau)=a(\sinh\tau-\tau)+b, \quad (a \text{ and } b \text{ are constants}),$$

$$p(\tau)=(a/3)[\sinh\tau-8\sinh(\tau/2)+3\tau]-b. \tag{44}$$

For $b=0$, Eq. (44) is the well known equation of state for a completely degenerate Fermi gas [10].

(iii) If $k=2/3$, Eq. (34) also admits the solution

$$q'(w)=\tfrac{2}{3}(3-\sqrt{3q})(1-\sqrt{3q}). \tag{45}$$

Using Eq. (3), this becomes

$$(p+\mu)\frac{dq}{d\mu}=\frac{2}{3}q(3-\sqrt{3q})(1-\sqrt{3q}), \quad q=\frac{dp}{d\mu} \tag{46}$$

which yields

$$q=\frac{1}{3}\left(\frac{3-\sqrt{\mu-b}/a}{1-\sqrt{\mu-b}/a}\right)^2, \tag{47}$$

and

$$p=\frac{\mu-b}{3}\,\frac{1-9a(\mu-b)^{-1/2}}{1-a(\mu-b)^{-1/2}} \quad (a \text{ and } b \text{ are constants}). \tag{48}$$

This pressure law (with $b=0$) has been derived by Tolman in searching for analytical solutions to Einstein's gravitational field equations. In that context it has proved to be somewhat helpful, since for large values of μ its approximate form is $\mu-3p=\text{const}\times\mu^{1/2}$ which is that for a highly compressed Fermi gas [11,12].

In Table I the Lie symmetry vector fields associated with the foregoing equations of state are listed.

V. SIMILARITY SOLUTIONS

(i) When $q=(\gamma-1)$ the infinitesimal generators are

$$W=A\theta+B, \quad \Theta=\frac{Aw}{\gamma-1}+D,$$

$$g\Psi=\left[A\,\frac{\gamma-2}{2(\gamma-1)}\,\theta+C\right]\Psi.$$

(1) For $A=1$, $B=D=0$, and C arbitrary, the characteristic equation reads

$$\frac{dw}{\theta}=\frac{d\theta}{\frac{w}{\gamma-1}}=\frac{d\Psi}{\left[\frac{\gamma-2}{2(\gamma-1)}\,\theta+C\right]\Psi}, \tag{49}$$

from which we get the similarity variable

$$\sigma=\theta^2-\frac{w^2}{\gamma-1}, \tag{50}$$

and the similarity solution

$$\Psi=\left(\theta+\frac{w}{\sqrt{\gamma-1}}\right)^{C\sqrt{\gamma-1}}\exp\left[\frac{\gamma-1}{2(\gamma-1)}\,w\right]F(\sigma,C). \tag{51}$$

If we make Eq. (51) satisfy Eq. (14) we obtain the ordinary differential equation in $F(z,C)$

$$\sigma F_{\sigma\sigma}(\sigma,C)+[1+C\sqrt{\gamma-1}]F_\sigma(\sigma,C)+\frac{(\gamma-2)^2}{16(\gamma-1)}F(\sigma,C)=0, \tag{52}$$

which transforms into a Bessel's equation

$$\xi^2 f''(\xi)+\xi f'(\xi)+(\xi^2-n^2)f=0, \quad n=C\sqrt{\gamma-1} \tag{53}$$

by setting

TABLE I. Lie symmetry vector fields of Eq. (14).

Sound speeds	Symmetry vector fields for (1) barotropic, (2) Fermi, and (3) Tolman flows
(1) $q=\gamma-1$	$L_A=\theta\frac{\partial}{\partial w}+\frac{w}{\gamma-1}\frac{\partial}{\partial\theta}+\frac{\gamma-2}{2(\gamma-1)}\theta\psi\frac{\partial}{\partial\psi}$, $L_B=\frac{\partial}{\partial\psi}$, $L_C=\psi\frac{\partial}{\partial\psi}$, $L_D=\frac{\partial}{\partial\theta}$. $[L_A,L_B]=-\frac{1}{1-\gamma}L_D$, $[L_A,L_D]=-L_B-\frac{\gamma-2}{2(\gamma-1)}L_C$, $[L_A,L_C]=[L_B,L_C]=[L_B,L_D]=[L_C,L_D]=0$.
(2) $q=\frac{1}{3}(1-e^{-2w})$	$L_A=\sqrt{q}\,\theta\frac{\partial}{\partial w}+\frac{\sqrt{3}}{2}\ln\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right)\frac{\partial}{\partial\theta}-\frac{\theta}{3\sqrt{q}}\psi\frac{\partial}{\partial\psi}$, $L_B=\sqrt{q}\frac{\partial}{\partial w}-\frac{1}{3\sqrt{q}}\psi\frac{\partial}{\partial\psi}$, $L_C=\psi\frac{\partial}{\partial\psi}$, $L_D=\frac{\partial}{\partial\theta}$. $[L_A,L_B]=-L_D$, $[L_A,L_D]=-L_B$, $[L_A,L_C]=[L_B,L_C]=[L_B,L_D]=[L_C,L_D]=0$.
(3) $\frac{1-\sqrt{3q}}{(3-\sqrt{3q})^3}=e^{-2w}$	$L_A=\sqrt{q}\,\theta\frac{\partial}{\partial w}+\frac{\sqrt{3}}{2}\ln\left(\frac{3-\sqrt{3q}}{1-\sqrt{3q}}\right)\frac{\partial}{\partial\theta}+\left(\sqrt{q}-\frac{2}{\sqrt{3}}\right)\theta\psi\frac{\partial}{\partial\psi}$, $L_B=\sqrt{q}\frac{\partial}{\partial w}+\sqrt{q}\psi\frac{\partial}{\partial\psi}$, $L_C=\psi\frac{\partial}{\partial\psi}$, $L_D=\frac{\partial}{\partial\theta}$. $[L_A,L_B]=-L_D$, $[L_A,L_D]=-L_B+\frac{2}{\sqrt{3}}L_C$, $[L_A,L_C]=[L_B,L_C]=[L_B,L_D]=[L_C,L_D]=0$.

$$F=\sigma^{-n/2}f(\xi),\quad \xi=\frac{\gamma-2}{2\sqrt{\gamma-1}}\sigma^{1/2}. \tag{54}$$

For n integer we have the linearly independent solutions

$$\Psi=\exp\left(-\frac{1-c_s^2}{2c_s^2}w\right)\left(\frac{\theta+\frac{w}{c_s}}{\theta-\frac{w}{c_s}}\right)^{n/2}\times J_n\left(-\frac{1-c_s^2}{2c_s}\sqrt{\theta^2-(w^2/c_s^2)}\right),$$

$$c_s=\sqrt{\gamma-1}. \tag{55}$$

As the Riemann invariants constant along the characteristic lines of Eqs. (9) and (10) are defined by [9]

$$R^{\pm}=\theta\pm\int\frac{dw}{c_s(w)}, \tag{56}$$

we see that Eq. (55) can be written as

$$\Psi=\exp\left(-\frac{1-c_s^2}{2c_s^2}w\right)\left(\frac{R^+}{R^-}\right)^{n/2}J_n\left(-\frac{1-c_s^2}{2c_s}\sqrt{R^+R^-}\right). \tag{57}$$

(2) For $A=D=0$, $B=1$, and C arbitrary, the characteristic equation reads

$$\frac{dw}{1}=\frac{d\theta}{0}=\frac{d\Psi}{C\Psi}. \tag{58}$$

The similarity variable and the similarity solution are, respectively,

$$z=\theta,\quad \Psi=e^{Cw}F(\theta,C) \tag{59}$$

with $F(\theta,C)$ satisfying the equation

$$F_{\theta\theta}(\theta,C)-C[(2-\gamma)+(\gamma-1)C]F(\theta,C)=0. \tag{60}$$

(ii) *The Fermi gas*. In this case the expressions for W, Θ, and $g\Psi$ are

$$W=\sqrt{q}(A\theta+B),\quad \Theta=A\frac{\sqrt{3}}{2}\ln\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right)+D, \tag{61}$$

$$g\Psi=\left(-\frac{A\theta+B}{3\sqrt{q}}+C\right)\Psi.$$

(1) For $A=1$, $B=D=0$, and C arbitrary the characteristic equation is

$$\frac{dw}{\theta\sqrt{q}}=\frac{d\theta}{\frac{\sqrt{3}}{2}\ln\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right)}=\frac{d\Psi}{\left(-\frac{\theta}{3\sqrt{q}}+C\right)\Psi}. \tag{62}$$

The similarity variable is

$$\sigma=\theta^2-\varphi^2,\quad \varphi=\frac{\sqrt{3}}{2}\ln\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right), \tag{63}$$

and the similarity solution reads

$$\Psi=\sqrt{(1/3q)-1}(\theta+\varphi)^C F(\sigma,C), \tag{64}$$

where $F(\sigma,C)$ satisfies the ordinary differential equation

$$\sigma F_{\sigma\sigma}(\sigma,C)+(1+C)F_\sigma(\sigma,C)+\frac{F(\sigma,C)}{12}=0. \tag{65}$$

If we set

$$F=\sigma^{-C/2}f(\xi),\quad \xi=\sqrt{\sigma/3}, \tag{66}$$

Eq. (65) becomes a Bessel's equation of order $n=C$

$$\xi^2 f''(\xi)+\xi f'(\xi)+(\xi^2-n^2)f(\xi)=0, \tag{67}$$

therefore, Ψ reads

$$\Psi=\left(\frac{1}{3q}-1\right)^{1/2}\left(\frac{\theta+\varphi}{\theta-\varphi}\right)^{C/2}J_n\left(\sqrt{\tfrac{1}{3}(\theta^2-\varphi^2)}\right). \tag{68}$$

The expressions for $\theta\pm\varphi$ appearing in Eq. (68) are nothing more than the Riemann invariants

$$R^{\pm}=\theta\pm\frac{\sqrt{3}}{2}\ln\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right). \tag{69}$$

(2) For $A=D=0$, $B=1$, and C arbitrary, the characteristic equation reads

$$\frac{dw}{\sqrt{q}}=\frac{d\theta}{0}=\frac{d\Psi}{\left(-\frac{1}{3\sqrt{q}}+C\right)\Psi}.$$

The similarity variable and the similarity solution read, respectively,

$$\sigma=\theta,\quad \Psi=\left(\frac{1}{3q}-1\right)^{1/2}\left(\frac{1+\sqrt{3q}}{1-\sqrt{3q}}\right)^{(\sqrt{3}/2)C}F(\theta,C),$$

with $F(\theta,C)$ satisfying the equation

$$F_{\theta\theta}(\theta,C)-(C^2-1/3)F(\theta,C)=0.$$

(iii) When one considers the Tolman pressure law one has

$$W=\sqrt{q}(A\theta+B),\quad \Theta=A\frac{\sqrt{3}}{2}\ln\left(\frac{3-\sqrt{3q}}{1-\sqrt{3q}}\right)+D,$$

$$g\Psi=\left[\left(\sqrt{q}-\frac{2}{\sqrt{3}}\right)(A\theta+B)+C\right]\Psi.$$

(1) For $A=1$, $B=D=0$, and C arbitrary, the characteristic equations are

$$\frac{dw}{\sqrt{q}\theta}=\frac{d\theta}{\frac{\sqrt{3}}{2}\ln\left(\frac{3-\sqrt{3q}}{1-\sqrt{3q}}\right)}=\frac{d\Psi}{\left[\left(\sqrt{q}-\frac{2}{\sqrt{3}}\right)\theta+C\right]\Psi}.$$

The integration of the first equation yields the similarity variable

$$\sigma=\theta^2-\varphi^2,\quad \varphi=\frac{\sqrt{3}}{2}\ln\left(\frac{3-\sqrt{3q}}{1-\sqrt{3q}}\right).$$

The similarity solution is

$$\Psi=\sqrt{(3-\sqrt{3q})(1-\sqrt{3q})}(\theta+\varphi)^C F(\sigma,C),$$

where $F(\sigma,C)$ satisfies the ordinary differential equation

$$\sigma F_{\sigma\sigma}(\sigma,C)+(1+C)F_\sigma(\sigma,C)+\frac{F(\sigma,C)}{12}=0,$$

with the positions (66) we obtain

$$\Psi=\sqrt{(3-\sqrt{3q})(1-\sqrt{3q})}\left(\frac{\theta+\varphi}{\theta-\varphi}\right)^{C/2}J_n\left(\sqrt{\frac{\theta^2-\varphi^2}{3}}\right),$$

where J_n is the Bessel's function of order $n=C$ (n integer). The Riemann invariants this time read

$$R^{\pm}=\theta\pm\frac{\sqrt{3}}{2}\ln\left(\frac{3-\sqrt{3q}}{1-\sqrt{3q}}\right)=\theta\pm\varphi;$$

so that the similarity solution can also be written as

$$\Psi=\sqrt{(3-\sqrt{3q})(1-\sqrt{3q})}\left(\frac{R^+}{R^-}\right)^{C/2}J_n\left[\left(\frac{R^+R^-}{3}\right)^{1/2}\right].$$

(2) For $A=D=0$, $B=1$, and C arbitrary, the characteristic equations read

$$\frac{dw}{\sqrt{q}}=\frac{d\theta}{0}=\frac{d\Psi}{(\sqrt{q}+C)\Psi}.$$

The similarity variable and the similarity solutions are, respectively,

$$\sigma=\theta,\quad \Psi=\frac{(3-\sqrt{3q})^{(\sqrt{3}C+3)/2}}{(1-\sqrt{3q})^{(\sqrt{3}C+1)/2}}F(\theta,C)$$

where $F(\theta,C)$ satisfies the ordinary differential equation

$$F_{\theta\theta}(\theta,C)-\tfrac{1}{3}(C+\sqrt{3})(3C+\sqrt{3})F(\theta,C)=0.$$

VI. CONCLUSIONS

A well known mathematical technique to obtain explicit solutions consists of finding the Lie groups which leave the equation under consideration invariant. In particular, the equation describing a relativistic flow in the hodograph plane contains an ''arbitrary element,'' the sound speed $c_s(w)$, so that the conditions for the existence of Lie symmetries are expressed by the additional Eqs. (32) and (33) connecting $q=c_s^2$ and w (the enthalpy density). Once these are solved we can determine all the possible forms of the sound speed which admit a four parameter goup of invariance. Then straightforward calculations permit one to obtain similarity solutions.

This procedure, which may appear more mathematically than physically motivated, gives some interesting and encouraging results in that the solutions of the additional equations lead to some well known pressure-energy density relations, i.e.: (1) the equation of state for a barotropic flow, (2) the Fermi pressure law for a completely degenerate relativistic gas, and surprisingly enough, (3) the Tolman equation of state with which Einstein's field equations also admit analytical solutions. The above mentioned pressure laws do not exhaust the solutions to Eqs. (32) and (33) and other physically interesting equations of state may be found by further analysis.

Moreover, we found that Eq. (14) admits an infinite group of symmetries in correspondence to the nontrivial pressure law (39) which satisfies the conditions for a real relativistic flow. In this case, Eq. (14) is integrable in an elementary way, while the system (9) and (10) can be decoupled into two inviscid Burger's equations. This result might be useful for testing numerical codes.

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